Euler's Constancy

Leonhard Euler is seldom remembered as one of the Enlightenment greats, but he should be. His discoveries changed the course of mathematics forever, and 300 years after his birth his ideas continue to resonate in classrooms and laboratories.

BY JOHN DERBYSHIRE

Who is the greatest mathematician of all time? In 1937, Eric Temple Bell, the most widely read historian and biographer of mathematics, placed Archimedes, Isaac Newton, and Karl Friedrich Gauss at the top of the list, adding, "It is not for ordinary mortals to attempt to arrange [these three] in order of merit." This judgment, widely known among mathematicians, stirred a protest in 1997 from Charlie Marion and William Dunham in *Mathematics Magazine*. The protest was in eight stanzas of verse, of which the fourth and fifth read:

> Without the Bard of Basel, Bell,
> You've clearly dropped the ball.
> Our votes are cast for Euler, L.
> Whose *Opera* says it all.
>
> Six dozen volumes—what a feat!
> Profound and deep throughout.
> Does Leonhard rank with the elite?
> Of this there is no doubt.

Marion and Dunham were paying tribute to the mathematician Leonhard Euler (1707–83), one of the great yet little-known figures from Europe's Age of Enlightenment. Euler's discoveries continue to influence such disparate fields as computer networking, harmonics, and statistical analysis, and they did nothing less than transform pure mathematics. Children still learn Euler's lessons in school. It was Euler, for instance, who gave the name i to the square root of −1. To mark his tercentenary, admirers are holding symposiums, concerts, and a two-week Euler tour, which will stop in St. Petersburg and Berlin, the two cities where he spent his working life, as well as Basel, Switzerland, the city of his birth. There is even an Euler comic book, *A Man to Be Reckoned With,* in German and English editions.

Compared to Gauss and Newton, both of whom published sparingly, Euler was prolific. This makes the assignment of precedence somewhat subjective. But Archimedes and Newton can hardly be excluded from the top ranks. For sheer breadth and quality of mathematical thought, I believe most scholars would place Gauss ahead of Euler. It is a close call, though, and nobody would disagree that Euler ranks with the crème de la crème in mathematical excellence. So who was he?

Leonhard Euler was born April 15, 1707, into a German-speaking family (the name is pronounced "Oiler"). His father, Paul Euler, was a Calvinist clergyman, and Leonhard remained a firm, uncritical Calvinist his whole life, believing that all events were preordained

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Leonhard Euler's visual impairment is evident in this 1760 portrait by Emanuel Jakob Handmann. Euler's eyesight began to fail in the 1730s, but his vision problems did little to curb his immense productivity.

by God at the Creation. He once wrote a tract defending the truth of Revelation against Enlightenment skeptics. These beliefs did not make him a grim fatalist. To the contrary, he was a cheerful, industrious, and kind-hearted man, reliably humble despite his fame. Though given to "good-natured sarcasm," as a contemporary noted, and short-lived outbursts of temper, he was altogether one of the more attractive personalities in the history of mathematics.

When the precocious 13-year-old Euler commenced his studies at Basel University in 1720, Johann Bernoulli, another great name in the history of numbers, held the chair of mathematics. Bernoulli was also an old acquaintance of Leonhard's father. Though a proud and prickly man with no great fondness for teaching, Bernoulli granted the boy individual seminars on Saturday afternoons, and must soon have recognized his mathematical ability. Paul Euler wanted his son to study theology and follow him into the clergy, but Bernoulli persuaded Rev. Euler to approve a switch to math and physics. Leonhard graduated in 1726, and published his first mathematical paper that same year.

At just that time, Johann Bernoulli's eldest son, Nicholas, died in St. Petersburg. Both Nicholas and his brother Daniel had taken positions at the new St. Petersburg Academy, established in 1724 as part of Tsar Peter the Great's grand plan to modernize his nation, and Daniel wrote to his father to suggest that Nicholas be succeeded by Leonhard Euler. Glad of the rare opportunity to attain an academician's post at such a young age, Euler traveled to St. Petersburg in May 1727, a month after his 20th birthday, and a month and a half after the death of Newton. Unfortunately, Peter the Great was already dead, and his wife and successor died just as Euler arrived. The new regime was skeptical of the academy, so the scholars took pains to make themselves appear useful to the state. Euler secured a commission in the imperial navy, though he seems never to have gone to sea.

Through the 1730s, Euler worked on various projects for the Russian state—notably in the areas of cartography and shipbuilding—while making his international reputation as a mathematician. These were unhappy years for Russia, with the country descending into state terror during the reign

of Empress Anna (1730–40). "Common prudence forced [Euler] into an unbreakable habit of industry," E. T. Bell writes, suggesting that Euler's extraordinary productivity had its foundations in this period. Another biographer remarks, "In all of Euler's vast correspondence there is no mention of politics." His Russian experiences either inoculated Euler against politics or confirmed an innately apolitical disposition.

In 1733, after Daniel Bernoulli left Russia in disgust at the continuing political horrors, Euler was elevated to the St. Petersburg Academy's chair of mathematics. Two years later, he made his name throughout Europe by solving the famous Basel Problem: finding a closed form—a precise value—for the infinite sum

$$1+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+\frac{1}{5^2}+\frac{1}{6^2}+\ldots$$

The Basel problem had already defeated many of the top mathematicians of Euler's time, including Jacob Bernoulli and Gottfried Leibniz, but Euler showed that the sum was $\pi^2/6$. It was a striking result. π (pi) is, of course, a well-known geometric constant, the ratio of a circle's circumference to its diameter. Mathematicians nowadays are accustomed to seeing it crop up in unexpected places, but in 1735 it seemed remarkable for such a geometric value to appear in the solution to a mathematical problem. It was Euler, by the way, who popularized the symbol π in its now-familiar usage.

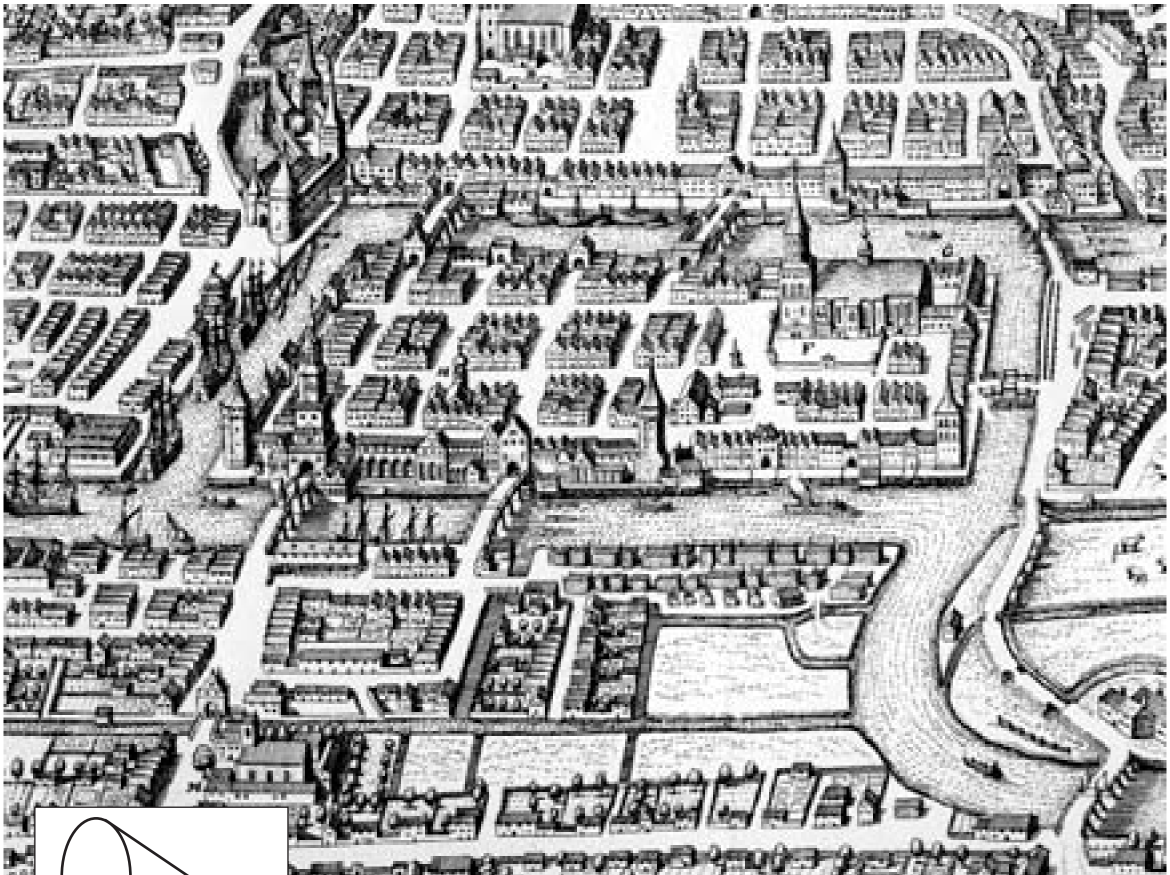
With the improvement in his finances that came with the mathematics professorship, Euler could afford to marry, and he took as his wife Katharina Gsell, the daughter of a Swiss painter whom Peter the Great had invited to his court. He and Katharina had 13 children in their 40 years together. Although only five of those children survived to adulthood, they managed to produce a large number of grandchildren—26 were alive at the time of Euler's death. Family life seems to have suited the great mathematician. Euler boasted that he could write mathematical papers with an infant on his knee—a claim that would be impressive even for a writer who traded only in words.

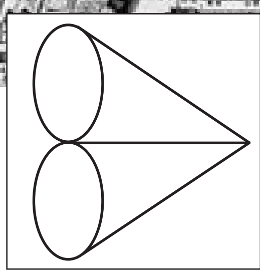
In the same year he triumphed over the Basel Problem, Euler suffered a severe fever that almost killed him. It was at this time, or soon afterward, that troubles with his eyesight began. By 1740, he had completely lost the sight in one eye. But failing vision did little to impair Euler's remarkable mathematical productivity. It was in this period of the later 1730s that he produced some of his best-known work. In 1736, for example, he published a famous paper solving the problem of the bridges of Königsberg (see opposite page). The central part of that Prussian city (now Russian and called Kaliningrad) was on a large island, where two branches of the Pregel River diverged. Seven bridges connected the city's various parts, and there was much speculation by the citizens as to whether one could make a complete tour of the town and return to the starting point, crossing each of the seven bridges exactly once. Euler proved that this could not be done. His paper, in which he developed a formula for determining the number and layout of different routes from one point to another, is regarded as the beginning of modern graph theory, a major branch of mathematics critical to the design of modern networks and circuits.

Also dating from this period is Euler's modestly titled paper "Various Observations About Infinite Series" (1737). Here, Euler considered an infinite sum generalized from the one shown for the Basel problem, with the exponent—i.e., *2*—in the denominators replaced by *any exponent at all*—i.e., x. Euler showed that this sum is equal to an elegant expression involving all the prime numbers (which can be divided only by 1 and themselves), of which there are an infinite number.

It was not only pure mathematical results that emerged from these fruitful years. Euler's book *Mechanica* (1736) recast Newton's theories of motion in the latest, most sophisticated mathematical language, allowing mechanical problems to be understood and solved in a less theoretical, and hence more practical, fashion. Applying findings from his twice-daily astronomical observations at the St. Petersburg observatory, Euler published a paper on calculating the precise instant of true noon. He won the prestigious Grand Prize of the Paris Academy in 1738 for a treatise on the nature of fire, and again in 1740 for a paper on the tides.

By 1741, Euler had had enough of Russia. Frederick the Great was on the throne of Prussia, and was intent on making his country a great European military and cultural power. He invited the 34-year-old Euler to be director of mathematics at the revivified Academy of Sciences in Berlin. Euler remained at this post for 25 years, through all the horrors of the Seven Years' War, when foreign armies occupied Berlin twice and one in 10 of Frederick's subjects died of hunger or disease, or by the sword.





In 1736, Euler solved a puzzle that had long stumped the people of Königsberg. He used a diagram similar to this (left) to demonstrate the impossibility of touring to all parts of the city, including the island section in the center, and returning to the starting point by crossing each of the city's seven bridges only once. Euler's diagram represented each bridge with a line, and enabled him to prove that every land portion of the city would have to have an even number of bridges in order to achieve the objective. His work is regarded as the beginning of graph theory, essential in the development of circuits in the computer age.

Euler's productivity never faltered. His greatest mathematical work of this period was the 1748 masterpiece *Introduction to Analysis of the Infinite.* "Analysis" is a key word in modern math. It names, in fact, all of that part of math that depends on the idea of a finite result emerging from some infinite process: The limits of infinite sequences, infinite sums and products, all of calculus and the classical theory of functions—this is "analysis" as the word is now used. It was the *Introduction,* more than anything else, that turned the meaning of the word toward this modern sense. Until Euler's time, analysis had been loosely used as a synonym for algebra. The *Introduction* has a good claim to being what math historian Carl Boyer called it: "the foremost [mathematics] textbook of modern times."

Aside from helping to establish modern analysis, Euler obtained one of the founding results of modern topology, a formula relating the number of vertices, edges, and faces of any flat-sided and simple (no "doughnut holes"!) solid figure: $V - E + F = 2$.

In applied mathematics, too, Euler made major contributions. He carried out some straightforwardly practical projects—designing a system of water pumps for the fountains at Frederick's Sans-Souci palace, for example. He also weighed in on the controversy over the nature of light, taking the side of the wave theorists against the Newtonians, who believed that light consisted of particles. The sun, wrote Euler, was "a bell ringing out light."

He viewed light as a vibration in the ether, analogous to sound waves propagating through air. (Twentieth-century discoveries indicate that light exhibits properties of *both* waves and particles.)

Acoustics was a long-standing interest; one of Euler's earliest papers, written at age 19 before he left Basel for St. Petersburg, dealt with the nature of sound. Euler's interest in acoustics no doubt emerged from the great pleasure he took in listening to music. In the profusion of works he published during the fertile period of the late 1730s when he was losing his eyesight was a treatise on musical theory. His disciple Nicolas Fuss observed in a eulogy that Euler was attempting nothing less than to find "the fountainhead of pleasurable harmonies." This 1739 work contained "too much geometry for the musician and too much music for the geometer," as Fuss said. Yet Euler's invention of the tonnetz—a two-dimensional lattice diagram for showing the relationships between musical notes and intervals—was a breakthrough in music theory, and is taught in advanced musicology courses to this day. Scholars such as Princeton musicologist Dmitri Tymoczko have recently advanced Euler's theory into spaces of more than two dimensions, opening up the possibility that entirely new musical forms, as pleasing to the ear as those of the Baroque and classical composers, might be out there in these abstract spaces waiting to be discovered.

Frederick the Great was himself a keen music lover, holding evening concerts of chamber music at his court that included such luminaries as Johann Sebastian Bach. But music may have been the only affinity between Frederick and his famous mathematician. The king wanted his court to be one of Europe's great salons, filled with brilliant people saying brilliant things. Euler, though well educated in philosophy, history, and literature—his phenomenal memory "did not permit him to forget anything," said the French philosophe Condorcet—held only commonplace opinions on those subjects. In a court that hosted Voltaire for several years, the plain, unsophisticated mathematician hardly could have shone. Soon after arriving in Berlin from St. Petersburg, he explained his reserve to Frederick's mother by saying, "Madame, I have come from a country where every person who speaks is hanged." His simple piety irritated Frederick, who was scornful of all religion. Relations between the king and his mathematician soured, Frederick referring cruelly to the partly blind Euler as "my Cyclops."

"I HAVE COME FROM a country where every person who speaks is hanged," the reserved Euler once said.

Incredibly, Euler's position at St. Petersburg had been kept open for him, and in 1766, four years after Catherine the Great took the throne of Russia, he packed his bags and returned, living the rest of his days there.

This period in St. Petersburg was autumnal for Euler. His one seeing eye deteriorated, and by 1771 he was totally blind, forced to rely on his son Johann and younger mathematicians such as Nicolas Fuss to serve as amanuenses.

In that year, a fire destroyed his house. Katharina died two years later. (Euler later married her 53-year-old half-sister, Salome.) Of his five children who lived to adulthood, the two daughters died, in 1780 and 1781.

None of this seems to have slowed Euler. From this last phase of his life came a three-volume work on optics; a textbook on algebra that was still used in American schools in the 1830s; and books on hydrodynamics, integral calculus, and insurance. In his pure-mathematical inquiries he ventured into uncharted territory by carrying the concept of function, which is central to analysis, into the realm of complex numbers. A function relates each of the inputs of an equation to only one output. If, for instance, y is the number of feet traversed by an object falling freely and x is the number of elapsed seconds on a stopwatch, then the formula $y = 16x^2$ expresses the number represented by y as a function of the number represented by x. Permitting the use of the square root of -1 as x or y opens up a rich field of inquiry, which came to its full flowering only in the mid-19th century, decades after Euler's death.

Also from this period dates the publication of Euler's unexpected pop-science bestseller *Letters to a German Princess* (1768). The source material here was more than 115 lessons in math, physics, and philosophy that Euler had written at Frederick the Great's command for the monarch's niece. *Letters* covered a remarkable range of topics: "Of the 12 tones of the harpsichord," "Of the azure color of the heavens," "Of moral and physical evil," and so on. Not all the science is strictly correct—one commentator sniffed that "Euler's strength lay rather in pure than in applied mathematics"—but *Letters* was immensely popular, and was translated into nine European languages.

There never was a mathematician as productive as Euler. Math writer W. W. Rouse Ball computed that from 1736, when Euler began publishing regularly, to his death from a stroke in 1783,

> there is for each and every fortnight in 47 years a separate effort of mathematical invention, digested, arranged, written in Latin, and amplified, often to a tedious extent, by corollaries and scholia. Through all this mass, the power of the inventor is almost uniformly distributed, and apparently without effort. There is nothing like this, except this, in the history of science.

Though it seems almost impertinent to emphasize any of the man's contributions above others, probably most mathematicians would agree that Euler's work in analysis advanced mathematics the furthest. It is here that his single most memorable result belongs. The famous Euler equation

$$e^{i\pi} + 1 = 0$$

manages to establish a correlation among five of the most important numbers (0, 1, *i*, *e*, and π—the last three all owe their symbols to Euler!) as well as among three key operations (addition, multiplication, and exponentiation).

Euler, uniquely among mathematicians, has not one but *two* "pure" numbers named after him. (Pure numbers aren't very relevant to the average person toting up his tax bill, but they are extremely significant in various kinds of mathematical work.) To 16 significant figures, these numbers are 2.718281828459045 and 0.5772156649015328. The first was glimpsed shortly after the invention of logarithms in the early 17th century, and employed in mathematical work by Jacob Bernoulli, Johann's elder brother, and by John Napier, but it was Euler who first showed its full importance: It is the basis of the exponential function for which growth rates are the same as the values at any given point on a graph. He also assigned it the symbol by which it has ever since been known: *e*. (This was not vanity on Euler's part; *e* was simply the first vowel not in common use for any mathematical purpose.)

The second number, known as "Euler's constant" and always denoted by γ (Greek gamma), turned up in Euler's explorations of logarithms during his early St. Petersburg days. It is the limit, as *n* becomes indefinitely large, of

$$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \ldots + \frac{1}{n} - \log_e n$$

Because the constant distills a complex calculation "sufficiently accurately and with very little effort," to use Euler's own words, it remains an invaluable tool in analytic number theory.

In addition to these particular numbers, Euler has an infinite series of integers named after him: the Euler numbers $E_0, E_2, E_4, E_6, \ldots$ (the odd-numbered ones are all zero). The first six in the series have values of 1, –1, 5, –61, 1,385, and –50,521. These numbers appear in certain problems in analysis and number theory.

Most mathematicians would die happy knowing a single theorem had been named after them. To have *numbers* associated with your name is an honor bestowed on very few.

Fuss's funeral eulogy paid tribute to Euler as "a good husband, good friend, good citizen, and loyal in all of his relations to society." That is more than elegiac boilerplate. All accounts of Euler's life suggest that he was an admirable man, generous not only to his family and friends but to his critics and rivals as well. When a dispute arose over precedence in what is now known as the Euler-Maclaurin method for computing infinite sums, Euler wrote to a friend, "I have very little desire for anything to be detracted from the fame of the celebrated Mr. Maclaurin since he probably came upon the same theorem for summing series before me, and . . . deserves to be named as its first discoverer."

That was Leonhard Euler: a mathematician of towering genius who lived nobly, calmly, cheerfully, and well. Perhaps his unassuming nature is one reason that the nonmathematical public does not better know his name. Let us hope this year's tercentenary celebrations will put matters right. ■